Superposition Principle and Wronskian Determinants Differential Equations X. Du

- **Linear independence** of functions
  - Two functions y_1 and y_2 are defined as **linearly independent** if y_1 and y_2 are not linear multiples of each other – i.e. there exist no constants c_1 and c_2 such that $c_1 y_1 = c_2 y_2$ given that $(c_1, c_2) \in \{\mathbb{R}^2 | (c_1, c_2) \neq (0,0)\}$.
  - A set of functions $S = \{y_1, y_2, ..., y_n\}$ is **linearly independent** if for any subset $\{f, g\} \subseteq S$, f and g are linearly independent.
- **Superposition Principle**: Given a linear homogeneous differential equation, any **linear combination** of solutions is also a solution to the differential equation.
  - Example: second-order. Let y_1 and y_2 be solutions to a linear homogeneous second-order differential equation $y'' + P(x)y' + Q(x)y = 0$. Then $y = c_1 y_1 + c_2 y_2$ (a linear combination of the solutions) is a solution to the differential equation.
  - In other words: Suppose that $y_1, y_2, \ldots, y_n$ are all solutions of a linear homogeneous differential equation. Then, $y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$ is a solution.
  - Why is the superposition principle useful? To find general solutions of linear homogeneous ordinary differential equations.
- **Wronskian Determinant**. Suppose there is a set of n functions $\{y_1, y_2, ..., y_n\}$, each of which are $n-1$ times differentiable on some interval $I \subseteq \mathbb{R}$. The **Wronskian Determinant** of the functions in this set is: $W(y_1, y_2, ..., y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$
  - If $W(x_o) \neq 0$ for some $x_o \in I$, then $\{y_1, y_2, ..., y_n\}$ is linearly independent on *I*.
  - If $\{y_1, y_2, ..., y_n\}$ is linearly dependent on *I*, then $W(x) = 0$ for all $x \in I$.
- **Wronskian Determinant** of a second-order linear homogeneous ODE.
  - Let the general solution of the differential equation be $y = c_1 y_1 + c_2 y_2$.
  - $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$
  - An IVP is solvable if $W \neq 0$.
  - If y_1 and y_2 are solutions to the differential equation, then either $W(y_1, y_2) = 0$ for all *x*, or $W(y_1, y_2) \neq 0$ for any *x*.
- How do we know that an IVP is solvable?
  - An IVP given initial conditions around x_o is solvable if $W(x_o) \neq 0$